

ON MINIMAL NORMS ON M_n MADJID MIRZAVAZIRI¹ AND MOHAMMAD SAL MOSLEHIAN²

ABSTRACT. In this note, we show that for each minimal norm $N(\cdot)$ on the algebra M_n of all $n \times n$ complex matrices, there exist norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on \mathbb{C}^n such that

$$N(A) = \max\{\|Ax\|_2 : \|x\|_1 = 1, x \in \mathbb{C}^n\}$$

for all $A \in M_n$. This may be regarded as an extension of a known result on characterization of minimal algebra norms.

1. INTRODUCTION

Let \mathcal{M}_n denote the algebra of all $n \times n$ complex matrices A with entries in \mathbb{C} together with the usual matrix operations. By an algebra norm (or a matrix norm) we mean a norm $\|\cdot\|$ on \mathcal{M}_n such that $\|AB\| \leq \|A\|\|B\|$ for all $A, B \in \mathcal{M}_n$. It is easy to see that the norm $\|A\|_\sigma = \sum_{i,j=1}^n |a_{ij}|$ is an algebra norm, but the norm $\|A\|_m = \max\{|a_{i,j}| : 1 \leq i, j \leq n\}$ is not an algebra norm, see [2].

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on \mathbb{C}^n . Then the norm $\|\cdot\|_{1,2}$ on \mathcal{M}_n defined by $\|A\|_{1,2} := \max\{\|Ax\|_2 : \|x\|_1 = 1\}$ is called the generalized induced (or g-ind) norm constructed via $\|\cdot\|_1$ and $\|\cdot\|_2$. If $\|\cdot\|_1 = \|\cdot\|_2$, then $\|\cdot\|_{1,1}$ is called an induced norm.

It is known that $\|A\|_C = \max\{\sum_{i=1}^n |a_{i,j}| : 1 \leq j \leq n\}$, $\|A\|_R = \max\{\sum_{j=1}^n |a_{i,j}| : 1 \leq i \leq n\}$ and the spectral norm $\|A\|_S = \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^*A\}$ are induced by ℓ_1, ℓ_∞ and ℓ_2 , respectively; cf. [3]. Recall that the ℓ_p -norm ($1 \leq p \leq \infty$) on \mathbb{C}^n is defined by

$$\ell_p(x) = \ell_p\left(\sum_{i=1}^n x_i e_i\right) = \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} & 1 \leq p < \infty \\ \max\{|x_1|, \dots, |x_n|\} & p = \infty \end{cases}$$

2000 *Mathematics Subject Classification.* Primary 15A60; Secondary 47A30, 46B99.

Key words and phrases. Induced norm; generalized induced norm; algebra norm; the full matrix algebra; minimal norm.

It is known that the algebra norm $\|A\| = \max\{\|A\|_C, \|A\|_R\}$ is not induced and it is not hard to show that it is not g-ind too; cf. Corollary 3.2.6 of [1].

A norm $N(\cdot)$ on M_n is called minimal if for any norm $\|\cdot\|$ on M_n satisfying $\|\cdot\| \leq N(\cdot)$ we have $\|\cdot\| = N(\cdot)$. It is known [1, Theorem 3.2.3] that an algebra norm is an induced norm if and only if it is a minimal element in the set of all algebra norms. Note that a generalized induced norm may not be minimal. For instance, put $\|\cdot\|_\alpha = \ell_\infty(\cdot)$, $\|\cdot\|_\beta = 2\ell_2(\cdot)$ and $\|\cdot\|_\gamma = \ell_2(\cdot)$. Then $\|\cdot\|_{\gamma,\beta} \leq \|\cdot\|_{\alpha,\beta}$ but $\|\cdot\|_{\gamma,\beta} \neq \|\cdot\|_{\alpha,\beta}$.

In [2], the authors investigate generalized induced norms. In particular, they examine the problem that “for any norm $\|\cdot\|$ on M_n , there are two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on \mathbb{C}^n such that $\|A\| = \max\{\|Ax\|_2 : \|x\|_1 = 1\}$ for all $A \in \mathcal{M}_n$?” In this short note, we utilize some ideas of [2] to study the minimal norms on \mathcal{M}_n . More precisely, we show that for each minimal norm $N(\cdot)$ on the algebra M_n of all $n \times n$ complex matrices, there exists norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on \mathbb{C}^n such that $N(A) = \max\{\|Ax\|_2 : \|x\|_1 = 1, x \in \mathbb{C}^n\}$ for all $A \in M_n$. In particular, if $N(\cdot)$ is an algebra norm, then $\|\cdot\|_1 = \|\cdot\|_2$. This may be regarded as an extension of the above known result on characterization of minimal algebra norms.

2. MAIN RESULT

For $x \in \mathbb{C}^n$ and $1 \leq j \leq n$, let $C_{x,j} \in \mathcal{M}_n$ be defined by the operator $C_{x,j}(y) = y_j x$. Hence $C_{x,j}$ is the $n \times n$ matrix with x in the j column and 0 elsewhere. Define $C_x \in \mathcal{M}_n$ by $C_x = \sum_{j=1}^n C_{x,j}$. Hence C_x is the $n \times n$ matrix whose all columns are x .

If $\|\cdot\|_{1,2}$ is a generalized induced norm on \mathcal{M}_n obtained via $\|\cdot\|_1$ and $\|\cdot\|_2$ then $\|C_x\|_{1,2} = \alpha\|x\|_2$, where $\alpha = \max\{|\sum_{j=1}^n y_j| : \|(y_1, \dots, y_j, \dots, y_n)\|_1 = 1\}$.

To achieve our goal, we need the following lemmas.

Lemma 2.1. [2, Theorem 2.7] *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on \mathbb{C}^n . Then $\|\cdot\|_{1,2}$ is an algebra norm on \mathcal{M}_n if and only if $\|\cdot\|_1 \leq \|\cdot\|_2$.*

Lemma 2.2. [2, Corollary 2.5] *$\|\cdot\|_{1,2} = \|\cdot\|_{3,4}$ if and only if there exists $\gamma > 0$ such that $\|\cdot\|_1 = \gamma\|\cdot\|_3$ and $\|\cdot\|_2 = \gamma\|\cdot\|_4$.*

Theorem 2.3. *Let $N(\cdot)$ be a minimal norm on \mathcal{M}_n , then $N(\cdot) = \|\cdot\|_{1,2}$ for some $\|\cdot\|_1$ and $\|\cdot\|_2$ on \mathbb{C}^n . Moreover, if $N(\cdot)$ is an algebra norm, then $\|\cdot\|_1 = \|\cdot\|_2$.*

Proof. For $x \in \mathbb{C}^n$, set

$$\|x\|_1 = \max\{N(C_{Ax}) : N(A) = 1, A \in \mathcal{M}_n\}$$

and

$$\|x\|_2 = N(C_x).$$

We shall show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on \mathbb{C}^n .

To see $\|\cdot\|_1$ is a norm, let $x \in \mathbb{C}^n$. Then $\|x\|_1 = 0$ if and only if $N(C_{Ax}) = 0$ for all matrix A with $N(A) = 1$, and this holds if and only if $Ax = 0$ for all A , or equivalently $x = 0$.

For $\alpha \in \mathbb{C}^n$ and $x, y \in \mathbb{C}^n$, we have

$$\begin{aligned} \|\alpha x\|_1 &= \max\{N(C_{A(\alpha x)}) : N(A) = 1, A \in \mathcal{M}_n\} \\ &= \max\{N(\alpha C_{Ax}) : N(A) = 1, A \in \mathcal{M}_n\} \\ &= \max\{|\alpha|N(C_{Ax}) : N(A) = 1, A \in \mathcal{M}_n\} \\ &= |\alpha| \max\{N(C_{Ax}) : N(A) = 1, A \in \mathcal{M}_n\} \\ &= |\alpha| \|x\|_1. \end{aligned}$$

and

$$\begin{aligned} \|x + y\|_1 &= \max\{N(C_{A(x+y)}) : N(A) = 1, A \in \mathcal{M}_n\} \\ &= \max\{N(C_{Ax} + C_{Ay}) : N(A) = 1, A \in \mathcal{M}_n\} \\ &\leq \max\{N(C_{Ax}) : N(A) = 1, A \in \mathcal{M}_n\} \\ &\quad + \max\{N(C_{Ay}) : N(A) = 1, A \in \mathcal{M}_n\} \\ &= \|x\|_1 + \|y\|_1. \end{aligned}$$

To see $\|\cdot\|_2$ is a norm, let $x \in \mathbb{C}^n$. Then $\|x\|_2 = 0$ if and only if $C_x = 0$ and this holds if and only if $x = 0$.

For $\alpha \in \mathbb{C}^n$ and $x, y \in \mathbb{C}^n$, we have

$$\|\alpha x\|_2 = N(C_{\alpha x}) = N(\alpha C_x) = |\alpha|N(C_x) = |\alpha| \|x\|_2.$$

and

$$\|x + y\|_2 = N(C_{x+y}) = N(C_x + C_y) \leq N(C_x) + N(C_y) = \|x\|_2 + \|y\|_2.$$

Now let $A \in \mathcal{M}_n \setminus \{0\}$. Then $N(\frac{A}{N(A)}) = 1$. So that

$$\|\frac{A}{N(A)}(x)\|_2 = \|N(C_{\frac{A}{N(A)}(x)})\|_2 \leq \|x\|_1$$

whence

$$\|Ax\|_2 \leq N(A)\|x\|_1.$$

Therefore $\|A\|_{1,2} \leq N(A)$. Since $N(\cdot)$ is a minimal norm, we conclude that $\|A\|_{1,2} = N(A)$.

If $N(A)$ is an algebra norm, then Lemma 2.1 implies that $\|\cdot\|_1 \leq \|\cdot\|_2$.

Next let $A \in \mathcal{M}_n$. It follows from $\|Ax\|_1 \leq \|A\|_{1,1}\|x\|_1 \leq \|A\|_{1,1}\|x\|_2$, ($x \in \mathbb{C}^n$) that $\|A\|_{2,1} \leq \|A\|_{1,1}$. In a similar fashion one can get

$$\|\cdot\|_{2,1} \leq \|\cdot\|_{k,k} \leq \|\cdot\|_{1,2} \quad (k = 1, 2).$$

By the minimality of $\|\cdot\|_{1,2}$ we deduce that $\|\cdot\|_{1,2} = \|\cdot\|_{1,1}$. It then follows from Lemma 2.2 that $\|\cdot\|_1 = \|\cdot\|_2$. \square

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